PHY V2500: QUANTUM MECHANICS I

Midterm Examination

October 21, 2025, 10:00 AM to 11:40 AM

Problem 1 (10 points)

Consider a very simple 2-state system defined by the Hamiltonian

$$H = \begin{pmatrix} E_0 & v \\ v & E_0 \end{pmatrix}$$

a) Find the eigenvalues and eigenvectors of this Hamiltonian.

b) At time $t = 0$, the system is in the state

$$\psi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

a) Find the state $\psi(t)$ at time t.

b) Find the inner product $\langle \psi(0) | \psi(t) \rangle$. (The absolute square of this will give the probability to find the system in the same state as we started with after time t. This problem is relevant for a number of physical situations, the associated production of K-mesons and neutrino oscillations being two examples from particle physics.)

Solution

a) The eigenvalue equation is given by

$$\begin{vmatrix} E_0 - \lambda & v \\ v & E_0 - \lambda \end{vmatrix} = 0$$

This simplifies as

$$(\lambda - E_0)^2 - v^2 = 0 \quad \Longrightarrow \quad \lambda = E_0 \pm v$$

The eigenvalue equation for the first eigenvalue $\lambda_1 = E_0 + v$ is

$$\begin{pmatrix} E_0 & v \\ v & E_0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = (E_0 + v) \begin{pmatrix} a \\ b \end{pmatrix} \quad \Longrightarrow \quad v\,a = v\,b \;\text{ or }\; b = a$$

For the second eigenvalue $\lambda_2 = E_0 - v$, we find $b = -a$. Thus the normalized eigenvectors are

$$\psi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \psi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

b) The given initial state can be written as

$$\psi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} (\psi_1 + \psi_2)$$

The state at time t is thus given by

$$\begin{aligned} \psi(t) &= \frac{1}{\sqrt{2}} \left(e^{-i(E_0+v)t/\hbar} \psi_1 + e^{-i(E_0-v)t/\hbar} \psi_2 \right) \\ &= e^{-iE_0 t/\hbar} \begin{pmatrix} \cos(vt/\hbar) \\ -i \sin(vt/\hbar) \end{pmatrix} \end{aligned}$$

c) The inner product $\langle \psi(0) | \psi(t) \rangle$ is given by

$$\langle \psi(0) | \psi(t) \rangle = (1,\ 0)\, e^{-iE_0 t/\hbar} \begin{pmatrix} \cos(vt/\hbar) \\ -i \sin(vt/\hbar) \end{pmatrix} = e^{-iE_0 t/\hbar} \cos(vt/\hbar)$$

The probability to find the state of eigenvalue $E_0 + v$ at time t is thus $| \langle \psi(0) | \psi(t) \rangle |^2 = \cos^2(vt/\hbar)$.

Problem 2 (10 points)

For the harmonic oscillator consider the wave function $x^2\, \psi_0(x)$ where $\psi_0(x)$ is the ground state wave function

$$\langle x | 0 \rangle = \left(\frac{m\omega}{\pi\hbar} \right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2}, \qquad \xi = \sqrt{\frac{m\omega}{\hbar}}\, x$$

We can write the given wave function as $\langle x |\, x^2\, | 0 \rangle$.

a) Write x in terms of a and $a^\dagger$ and show that $x^2\, \psi_0(x)$ can be written as a linear combination of two eigenstates of the oscillator. Identify these states and the precise linear combination.

b) Using these results obtain the wave function $\psi(x,t)$ at time t if the the wave function at time $t = 0$ is $\psi(x,0) = x^2\, \psi_0(x)$.

Solution

a) From the given formulae, adding a and $a^\dagger$, we find

$$x = \sqrt{\frac{\hbar}{2m\omega}}\, (a + a^\dagger)$$

Thus

$$x^2 = \frac{\hbar}{2m\omega} (a^2 + (a^\dagger)^2 + a a^\dagger + a^\dagger a)$$

This gives

$$\begin{aligned}\psi(x) &= x^2\psi_0 = \langle x|\, x^2\, |0\rangle = \frac{\hbar}{2m\omega}\langle x|\, a^2 + (a^\dagger)^2 + aa^\dagger + a^\dagger a\, |0\rangle \\ &= \frac{\hbar}{2m\omega}\langle x|\, (a^\dagger)^2 + aa^\dagger\, |0\rangle \\ &= \frac{\hbar}{2m\omega}(\langle x|0\rangle + \sqrt{2}\,\langle x|2\rangle)\end{aligned}$$

where we used $a\,|0\rangle = 0$, $aa^\dagger\,|0\rangle = a\,|1\rangle = |0\rangle$ and $a^\dagger a^\dagger\,|0\rangle = a^\dagger\,|1\rangle = \sqrt{2}\,|2\rangle$, which follow from the given action of a and $a^\dagger$.

I did not ask to normalize this wave function, but if you do, the normalized ψ is

$$\psi = \frac{1}{\sqrt{3}}(\langle x|0\rangle + \sqrt{2}\,\langle x|2\rangle)$$

This follows from the fact that $\langle x|0\rangle$ and $\langle x|2\rangle$ are orthonormal.

The state $|0\rangle$ has energy $\frac{1}{2}\hbar\omega$ while $|2\rangle$ has energy $(2+\frac{1}{2})\hbar\omega$. The state at time t is thus given by

$$\begin{aligned}\psi(x,t) &= \frac{\hbar}{2m\omega}\left(e^{-i\omega t/2}\,\langle x|0\rangle + e^{-i(2+\frac{1}{2})\omega t}\,\langle x|2\rangle\right) \\ &= \frac{\hbar}{2m\omega}e^{-i\omega t/2}\left(\langle x|0\rangle + e^{-i2\omega t}\,\langle x|2\rangle\right)\end{aligned}$$

If you used the normalized wave function this will be

$$\psi(x,t) = \frac{e^{-i\omega t/2}}{\sqrt{3}}\left(\langle x|0\rangle + e^{-i2\omega t}\,\langle x|2\rangle\right)$$

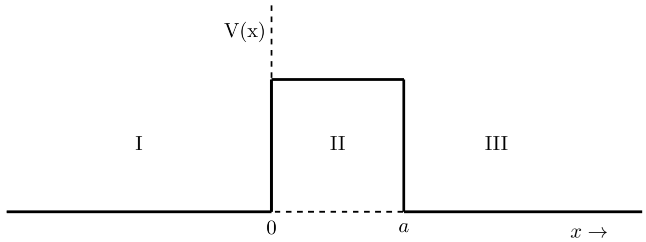
Problem 3 (10 points)

Consider the potential step in one dimension given by

$$V(x) = \begin{cases} 0 & -\infty < x < 0 \qquad \text{Region I} \\ V_0 & 0 < x < a \qquad \text{Region II} \\ 0 & a < x < \infty \qquad \text{Region III} \end{cases}$$

where V_0 is a positive constant. The potential is shown in the figure. This is the problem solved in class for the case of the energy $E < V_0$. Here we will consider $E > V_0$. Write down the Schrödinger equation and its solution in each of the three regions and the matching conditions needed to connect them. These should be expressed as matrix equations relating amplitudes in different regions.

(These can be solved for the reflected and transmitted amplitudes in terms of the incoming amplitude, but you do not have to do that.)



Solution

The Schrödinger equation for wave functions of energy E is

$$\frac{d^2\psi}{dx^2} = -\left[\frac{2m}{\hbar^2}(E-V)\right]\psi$$

Since $E > v_0$, $E - V$ is positive in all three regions, so we define

$$\begin{aligned} k &= \sqrt{\frac{2mE}{\hbar^2}} && \text{(Regions I, III)} \\ k' &= \sqrt{\frac{2m(E-V_0)}{\hbar^2}} && \text{(Region II)} \end{aligned}$$

The solution in each of the regions can be written as

$$\begin{aligned} \psi_{\mathrm{I}} &= A\,e^{ikx} + B\,e^{-ikx} \\ \psi_{\mathrm{II}} &= C\,e^{ik'x} + D\,e^{-ik'x} \\ \psi_{\mathrm{III}} &= E\,e^{ikx} + F\,e^{-ikx} \end{aligned}$$

The matching conditions are the continuity of the wave function and its first derivative across the interfaces. Thus at the interface $x = 0$ we find

$$\begin{aligned} \psi_{\mathrm{I}}(0) = \psi_{\mathrm{II}}(0) &\implies & A + B = C + D \\ \psi'_{\mathrm{I}}(0) = \psi'_{\mathrm{II}}(0) &\implies & ik(A-B) = ik'(C-D) \end{aligned}$$

Similarly at $x = a$ we find

$$\begin{aligned} \psi_{\mathrm{I}}(a) = \psi_{\mathrm{II}}(a) &\implies & Ce^{ik'a} + De^{-ik'a} = Ee^{ika} + De^{-ika} \\ \psi'_{\mathrm{I}}(a) = \psi'_{\mathrm{II}}(a) &\implies & ik'(Ce^{ik'a} - De^{-ik'a}) = ik(Ee^{ika} - De^{-ika}) \end{aligned}$$

Useful formulae

Integral

$$\int_0^\infty dx\, e^{-ax^2} = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

By differentiating both sides with respect to a, you can calculate integrals with additional factors of x^2, x^4, etc.

Basic commutation rules

$$\begin{aligned} \hat{x}_i \hat{x}_j - \hat{x}_j \hat{x}_i &= 0 \\ \hat{p}_i \hat{p}_j - \hat{p}_j \hat{p}_i &= 0 \\ \hat{x}_i \hat{p}_j - \hat{p}_j \hat{x}_i &= i\hbar\, \delta_{ij} \end{aligned}$$

Schrödinger equation

$$\begin{aligned} i\hbar \frac{\partial \psi}{\partial t} &= \left[-\frac{\hbar^2}{2m} \nabla^2 + V \right] \psi \\ \left[-\frac{\hbar^2}{2m} \nabla^2 + V \right] \psi &= E\psi \end{aligned}$$

Harmonic oscillator

$$\begin{aligned} H &= \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2 = \hbar\omega(a^\dagger a + \tfrac{1}{2}) \\ a &= \sqrt{\frac{m\omega}{2\hbar}}\, \hat{x} + \frac{i}{\sqrt{2m\hbar\omega}}\, \hat{p} \qquad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\, \hat{x} - \frac{i}{\sqrt{2m\hbar\omega}}\, \hat{p} \\ |n\rangle &= \frac{a^{\dagger n}}{\sqrt{n!}} |0\rangle\,, \qquad a\,|n\rangle = \sqrt{n}\; |n-1\rangle\,, \qquad a^\dagger\,|n\rangle = \sqrt{n+1}\; |n+1\rangle \\ H\,|n\rangle &= \hbar\omega(n + \tfrac{1}{2})\, |n\rangle \\ \langle x|n\rangle &= \left(\frac{m\omega}{\pi\hbar} \right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n\, n!}} H_n(\xi)\, e^{-\frac{1}{2}\xi^2}, \qquad \xi = \sqrt{\frac{m\omega}{\hbar}}\, x \\ H_0(\xi) &= 1, \qquad H_1(\xi) = 2\xi, \qquad H_2(\xi) = 4\xi^2 - 2 \end{aligned}$$